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Waves on Noncommutative Spacetimes

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1 Introduction

Studies on the formulation of physical theories on the Moyal plane were initiated in recent times by Doplicher et al. [1]. Interest in such algebras was also stimulated by the work of string theorists who encountered them in a certain decoupling limit [2].

The d-dimensional Groenewold-Moyal spacetime is an algebra $\mathcal{A}_\theta(\mathbb{R}^d)$ generated by elements $\hat{x}_\mu$ ($\mu \in [0, 1, 2, \cdots, d-1]$) with the commutation relation

$$[\hat{x}_\mu \,, \hat{x}_\nu] = i\theta_{\mu\nu}\mathbf{1} \,, \tag{1}$$

$\theta_{\mu\nu}$ being real constants antisymmetric in its indices. In the limit $\theta_{\mu\nu} = 0$, $\hat{x}_0$ and $\hat{x}_i$ are time- and space- coordinate functions. If $x = (x_0 \,, \vec{x})$ is a point of $\mathbb{R}^d$ when $\theta_{\mu\nu} = 0$, then

$$\hat{x}_0(x) = x_0 \,, \quad \hat{x}_i(x) = x_i \,. \tag{2}$$

Thus $\hat{x}_0$, $\hat{x}_i$ are operators in $\mathcal{A}_\theta(\mathbb{R}^d)$ which become time and space coordinate functions when $\theta_{\mu\nu} = 0$. There is an extensive literature on the formulation of quantum field theories ($qft's$) on $\mathcal{A}_\theta(\mathbb{R}^d)$ and on their phenomenology [3]. The focus of much of this work is

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on space-space noncommutativity ($\theta_{ij} \neq 0,\ i,j \in [1,2,...d-1]$). But it is time-space noncommutativity ($\theta_{0i} \neq 0$) with its implications for causality and foundations of quantum theory which leads to strikingly new physics. The formulation of unitary $qft's$ when $\theta_{0i} \neq 0$ is nontrivial and was carefully done already by Doplicher et al. [1].

In recent papers [4, 5], Balachandran et al. formulated unitary quantum mechanics when $\theta_{0i} \neq 0$ basing themselves on the ideas of Doplicher et al. Consequences of spacetime noncommutativity in the quantum mechanics of atoms and molecules have been explored by Balachandran and Pinzul [6].

Previous work on $\mathcal{A}_\theta(\mathbb{R}^d)$ was focused on the formulation of quantum theory. Effects of noncommutativity on classical waves and particles have largely remained untreated in particular when $\theta_{ij} \neq 0$ (see however [9]). In this paper we discuss "classical" waves on $\mathcal{A}_\theta(\mathbb{R}^d)$, assuming time-space noncommutativity ($\theta_{ij} = 0\,, \theta_{0i} \neq 0$).

The approach adopted in Doplicher et al. [1] and subsequently in Balachandran et al. [4] to study space-time noncommutativity is different from the string theory motivated studies in the literature due to Gomis and Mehen [7] and other authors [8], which found that field theories on noncommutative spacetime are perturbatively nonunitary. As explained in detail in Balachandran et al. [4], in the former approach, the amount τ of time translation is not "coordinate time", the eigenvalue of $\hat{x}_0$. For $\theta = 0$, these two could be identified, while for $\theta \neq 0$, $\hat{x}_0$ is an operator not commuting with $\hat{x}_1$, and cannot be interchanged with τ. The separation of eigenvalues of $\hat{x}_0$ from the amount of time translation is the central reason for the unitarity of the theories as formulated in Doplicher et al. and Balachandran et al [1, 4]. This is analogous to the situation in quantum mechanics, where if $\hat{p}$ is the momentum operator, spatial translation by amount ξ implemented by $\exp(i\xi\hat{p})$ is not the eigenvalue of the position operator $\hat{x}$.

In the algebraic approach (which is mandatory if $\theta_{\mu\nu} \neq 0$), waves are elements of the spacetime algebra. That is the case also for the commutative space-time $C^0(\mathbb{R}^d)$. The act of observation, such as measurement of mean intensity over the time interval T, is represented by a state on this algebra. In the following section, we describe this approach, valid equally for commutative and noncommutative algebras. Subsequently it is applied to interference for a double slit experiment for the algebra $\mathcal{A}_\theta(\mathbb{R}^d)$. In cases where a double image of a star is formed by a cosmic string, it causes interference as well which is affected by θ_{0i}. This phenomenon is examined in the final section.

Novel phenomena are observed in interference when $\theta_{0i} \neq 0$. For instance if the time of observation T is too small, then, as indicated before, one sees constant intensity and no interference on the screen. Interference returns for larger times, but it is shifted and deformed as a function of $\frac{\theta w}{T}$ ($\theta = \sqrt{\Sigma_i \theta_{0i}^2}\,, w$: frequency of the wave). The familiar interference pattern is recovered only when $\frac{\theta w}{T} \to 0$.

2 Classical Waves and Particles on Algebras

The algebra $\mathcal{A}_\theta(\mathbb{R}^d)$ has generators $\hat{x}_\mu$ with relations $[\hat{x}_\mu\,, \hat{x}_\nu] = i\theta_{\mu\nu}$, $\theta_{\mu\nu} = -\theta_{\nu\mu}$ being real constants. We assume that $\theta_{ij} = 0$, and orient $\vec{\theta}_0 = (\theta_{01}\,, \cdots \theta_{0d-1})$ in some direction $\vec{n}$.

Thus, for us,

$$[\hat{x}_i\,,\hat{x}_j]=0\,,\quad [\hat{x}_0\,,\hat{x}_i]=i\theta n_i\,,\quad \theta\in\mathbb{R},\quad \vec{n}\cdot\vec{n}=1\,. \tag{3}$$

We can set $\theta\geq 0$. This does not entail loss of generality since θ flips in sign when $\hat{x}_i\longrightarrow -\hat{x}_i$. So $\vec{\theta}_0=\theta\vec{n}$, $\theta\geq 0$.

For $\theta=0$, $\mathcal{A}_\theta(\mathbb{R}^d)$ is the algebra $C^0(\mathbb{R}^d)$ of functions on $\mathbb{R}^d$. We first outline the algebraic approach for $\theta=0$. It generalizes easily to $\theta\neq 0$.

i. Classical Theory on Commutative Algebra:

Let us first examine waves. They are fields $\hat{\psi}$ on spacetime so that $\hat{\psi}\in\mathcal{A}_0(\mathbb{R}^d)$. It is enough to consider scalar waves. Then $\hat{\psi}(x)$ for $x=(x_0\,,\vec{x})\in\mathcal{A}_0(\mathbb{R}^d)$ is the amplitude of the wave $\hat{\psi}(x)$ at x. It is the solution of a wave equation such as

$$\left(\partial_0^2-\sum_1^{d-1}\partial_i^2\right)\hat{\psi}(x)=0\,. \tag{4}$$

The algebra $\mathcal{A}_0(\mathbb{R}^d)$ contains not just $\hat{\psi}$, but functions of $\hat{\psi}$ as well. It is reasonable to assume that a general element $\hat{\alpha}\in\mathcal{A}_0(\mathbb{R}^d)$ has the Fourier representation

$$\hat{\alpha}=\int d^dk\,\tilde{\alpha}(k)\,e^{ik_0\hat{x}_0}e^{i\vec{k}\cdot\vec{\hat{x}}}\,, \tag{5}$$

where $\hat{x}_\mu$ are the coordinate functions: $\hat{x}_\mu(x)=x_\mu$.

We can measure many attributes of a wave. For example, we can measure its mean intensity over the time interval $[x_0-\frac{T}{2}\,,x_0+\frac{T}{2}]$. It is

$$I=\frac{1}{T}\int_{x_0-\frac{T}{2}}^{x_0+\frac{T}{2}}dx_0\,|\hat{\psi}(x_0\,,\vec{x})|^2\,. \tag{6}$$

We want to interpret this measurement as the application of a state on a particular element of the algebra since states are defined also for noncommutative algebras.

A state ω on a $*$-algebra $\mathcal{A}$ with unity $\mathbf{1}$ is a linear map [10],

$$\omega:\hat{\alpha}\in\mathcal{A}\longrightarrow\mathbb{C} \tag{7}$$

which is positive

$$\omega(\hat{\alpha}^*\hat{\alpha})\geq 0 \tag{8}$$

and normalized :

$$\omega(\mathbf{1})=1\,. \tag{9}$$

Thus states define probabilities and $\omega(\hat{\alpha})$ is the mean value of $\hat{\alpha}$.

Coming back to (6), for intensity, we associate the observable $\hat{I}\in\mathcal{A}_0(\mathbb{R}^d)$ where

$$\hat{I}=|\hat{\psi}|^2\,. \tag{10}$$

Measurement of the mean value of $\hat{\alpha} \in \mathcal{A}_0(\mathbb{R}^d)$ at $\vec{x}$ in the time-interval $[x_0 - \frac{T}{2}\,, x_0 + \frac{T}{2}]$ is represented by the state ω where

$$\omega(\hat{\alpha}) = \frac{1}{T} \int_{x_0 - \frac{T}{2}}^{x_0 + \frac{T}{2}} dx_0\, \hat{\alpha}(x_0\,, \hat{x})\,. \tag{11}$$

So ω depends on x_0, T and $\vec{x}$. Then

$$I = \omega(\hat{I})\,. \tag{12}$$

Thus to an observable, we assign an element $\hat{\alpha} \in \mathcal{A}_{\mathbf{0}}(\mathbb{R}^d)$ and to a measurement, a state ω on $\mathcal{A}_{\mathbf{0}}(\mathbb{R}^d)$. The result of this measurement of $\hat{\alpha}$ is $\omega(\hat{\alpha})$.

A classical particle too can be described by a similar formalism. Instead of working with $\mathbb{R}^d$, it is best to include momenta also and work with $\mathbb{R}^{2d-1}$. A point of $\mathbb{R}^{2d-1}$ is $(x_0, \vec{x}, \vec{p})$ where $\vec{p}$ denotes momenta. The algebra is then $\mathcal{A}_0(\mathbb{R}^{2d-1})$. If $\hat{\alpha} \in \mathcal{A}_0(\mathbb{R}^{2d-1})$, $\hat{\alpha}(x_0, \vec{x}, \vec{p})$ is the value of the observable $\hat{\alpha}$ at time x_0 for a particle with position $\vec{x}$ and momentum $\vec{p}$. For energy $\widehat{E} \in \mathcal{A}_0(\mathbb{R}^{2d-1})$ in a possibly time-dependent potential, we can have $\widehat{E}(x_0\,, \vec{x}, \vec{p}) = \frac{\vec{p}^2}{2m} + V(x_0\,, \vec{x})$.

States too can be introduced. For example, define ω by

$$\omega(\hat{\alpha}) = \frac{1}{T} \int_{x_0 - \frac{T}{2}}^{x_0 + \frac{T}{2}} dx_0\, \hat{\alpha}(x_0\,, \vec{x}\,, \vec{p})\,. \tag{13}$$

So ω depends on $x_0, T, \vec{x}$ and $\vec{p}$. $\omega(\widehat{E})$ is the mean energy in the time interval $[x_0 - \frac{T}{2}\,, x_0 + \frac{T}{2}]$ for a particle at $\vec{x}$ with momentum $\vec{p}$.

We, however, will not pursue point-particle theory any further.

ii. Classical Waves on Noncommutative Algebra $\mathcal{A}_\theta(\mathbb{R}^d)$

As remarked already, states can be defined also on $\mathcal{A}_\theta(\mathbb{R}^d)$. Thus to carry the discussion forward, we must identify waves in $\mathcal{A}_\theta(\mathbb{R}^d)$ say by wave equations, associate observables to waves and define suitable states. We will do so in the context of interference and diffraction for $d \leq 4$ in what follows. But we must emphasize one strikingly new feature of $\theta \neq 0$. Then since $\hat{x}_0$ and $\hat{x}_i$ do not commute, by the uncertainty principle, we can not simultaneously localize time in an interval T and sharply localise spatial coordinates. So a state like ω in (11) with exactly the same features does not exist for $\theta \neq 0$. We can at best approximate it.

We consider free massless scalar fields $\hat{\psi} \in \mathcal{A}_\theta(\mathbb{R}^d)$ for $d = 2, 3, 4$. Such massless scalar fields obey the standard wave equation

$$(\partial_0^2 - \vec{\nabla}^2)\hat{\psi} = 0 \tag{14}$$

for $\theta = 0$. We must find its analogue for $\theta \neq 0$.

For simplicity, we choose $\vec{n} = (1, 0, 0)$, if necessary by applying a spatial rotation on $\hat{x}_i$.

Let

$$\widehat{P}_0 = -\frac{1}{\theta} ad\, \hat{x}_1$$
$$ad\, \hat{\alpha}\ \hat{\beta} := [\hat{\alpha}\,, \hat{\beta}]\,. \tag{15}$$

Then

$$\widehat{P}_0 \hat{x}_0 = i\,, \quad \widehat{P}_0 \hat{x}_i = 0,\ i \geq 1\,. \tag{16}$$

So $\widehat{P}_0$ substitutes for $i\frac{\partial}{\partial x_0}$ and we can identify ∂_0 with $-i\widehat{P}_0$:

$$\partial_0 \longrightarrow -i\widehat{P}_0\,. \tag{17}$$

Similarly

$$\partial_1 \longrightarrow i\widehat{P}_1 = -\frac{i}{\theta} ad\, \hat{x}_0 \tag{18}$$

while

$$\partial_a \longrightarrow \partial_a := i\widehat{P}_a\,, \quad a = 2, 3, \tag{19}$$

∂_a in (19) being the conventional differentiations. So the noncommutative elementary massless wave equation is

$$(\widehat{P}_0^2 - \widehat{P}_1^2 - \widehat{P}_a^2)\hat{\psi} = 0\,. \tag{20}$$

It has plane wave solutions

$$\hat{\psi}_{\vec{k}} = e^{ik_i \hat{x}_i} e^{-iw\hat{x}_0}\,, \tag{21}$$

with a standard dispersion relation:

$$w^2 - \vec{k}^2 = 0\,. \tag{22}$$

The general solution is a superposition of plane waves.

We note that for electromagnetic (EM) waves, (20) receives corrections in powers of θ, since in noncommutative spacetimes, the EM Lagrangian gives nonlinear equations of motion. Interestingly enough monochromatic plane waves of the form (21) are solutions to the nonlinear equations of motion to every order in θ [11]. But their superposition is not. Nevertheless, as the inclusion of this effect will give only higher order corrections in θ to our results, they are not treated in this paper. Similarly the use of "covariant coordinates" (cf. [12] and references therein) will not affect leading order results in θ and hence we work with the standard noncommutative coordinates.

a) $d = 2$:

The problem we examine is the interference of two plane waves with the same frequency. It can be generalized, but several essential points are well illustrated by this example. Thus we consider

$$\hat{\psi} = \hat{\psi}_k + \hat{\psi}_{-k}\,,$$
$$\hat{\psi}_{\pm k} = e^{\pm ik\hat{x}_1} e^{-ik\hat{x}_0}\,, \quad k > 0\,. \tag{23}$$

We see that the intensity

$$\hat{I} = |\hat{\psi}|^2 \tag{24}$$

is an operator in $\mathcal{A}_\theta(\mathbb{R}^2)$.

It is not possible to achieve a state with a sharp localization in position and time. Instead, we look for a state ω with a reasonable spatial localization around a point x_1, it will be rather delocalized in time. We define $\omega \equiv \omega_\gamma$ in terms of a density matrix according to

$$\omega_\gamma(\hat{\alpha}) = \frac{Tr\hat{\gamma}\hat{\alpha}}{Tr\hat{\gamma}(\mathbf{1})}\,, \tag{25}$$

where

$$\hat{\gamma} = \hat{\psi}_T(\hat{x}_0 - x_0)\hat{\delta}(\hat{x}_1 - x_1)\,\hat{\psi}_T(\hat{x}_0 - x_0)\,. \tag{26}$$

Here by $\hat{\delta}$ and $\hat{\psi}_T$ we mean the following operators:

$$\hat{\delta}(\hat{x}_1 - x_1) \;=\; \frac{1}{2\pi}\int dk\, e^{ik(\hat{x}_1 - x_1)}\,, \tag{27}$$

$$\hat{\psi}_T(\hat{x}_0 - x_0) \;=\; \int_{-\frac{T}{2}}^{\frac{T}{2}} d\lambda\;\hat{\delta}(\hat{x}_0 - x_0 - \lambda)\,. \tag{28}$$

Let $|x_1'\rangle$ be the eigenstate of $\hat{x}_1$ for the eigenvalue x_1':

$$\hat{x}_1|x_1'\rangle = x_1'|x_1'\rangle\,. \tag{29}$$

Then

$$\hat{\delta}(\hat{x}_1 - x_1)|x_1'\rangle = \delta(x_1' - x_1)|x_1'\rangle\,, \tag{30}$$

where on the right hand side stands an ordinary delta function.

$\hat{\psi}_T$ is defined on eigenstates $|x_0'\rangle$ of $\hat{x}_0$:

$$\begin{aligned} \hat{x}_0|x_0'\rangle &= x_0'|\hat{x}_0\rangle\,, \\ \hat{\psi}_T(\hat{x}_0 - x_0)|x_0'\rangle &= \psi_T(x_0' - x_0)|x_0'\rangle\,. \end{aligned} \tag{31}$$

Here $\psi_T(\xi)$ is the characteristic function on the interval $[-\frac{T}{2}\,,\frac{T}{2}]$:

$$\psi_T(\xi) = \begin{cases} 1 & \text{for} \quad |\xi| < \frac{T}{2} \\ 0 & \text{for} \quad |\xi| > 0\,. \end{cases} \tag{32}$$

It is easy to show that $\hat{\gamma}$ is a positive operator and that ω_γ is a state. Now $\psi_T^2 = \psi_T$. Hence for $\theta = 0$, we can write $\hat{\gamma} = \delta(\hat{x}_1 - x_1)\psi_T(\hat{x}_0 - x_0)$. The corresponding $\omega_{\hat{\gamma}}$ describes an experiment at spatial location x_1 which is averaged uniformly over the time interval $[x_0 - \frac{T}{2}\,, x_0 + \frac{T}{2}]$. For $\theta \neq 0$, $\hat{\gamma}$ is an approximation to such an experiment.

But for $\theta \neq 0$, $\hat{\gamma}$ does not have sharp spatial localization. If it did, then for

$$\hat{\alpha} = \hat{\delta}(\hat{x}_1 - y_1)\,, \tag{33}$$

we should get

$$\omega(\hat{\alpha}) = \delta(x_1 - y_1)\,. \tag{34}$$

Instead we find (cf. Appendices 1 and 2)

$$\omega_\gamma(\hat{\alpha}) = \frac{2}{\pi}\frac{\theta}{T}\frac{1}{(x_1-y_1)^2}\sin^2\left[\frac{T(x_1-y_1)}{2\theta}\right] . \tag{35}$$

As $\frac{T}{\theta} \to \infty$, it approaches to $\delta(x_1 - y_1)$ as it should.

It is important to emphasize that in order to interpret the action of the state $\omega_{\hat{\gamma}}$ on $|\hat{\psi}|^2$ as the measurement of intensity at a given spatial location, say x_1, it is necessary to be able to localize x_1 with a reasonable precision which becomes sharply localized as $\theta \to 0$. $\omega_{\hat{\gamma}}$ with density matrix $\hat{\gamma}$ does this job perfectly: It approximately localizes the point where the measurement is taken (c.f. equation (35)) and it has the correct commutative limit.

A convenient way to calculate the traces such as those in (35) is to use the coherent states. Let

$$a = \frac{\hat{x}_0 + i\hat{x}_1}{\sqrt{2\theta}} , \quad a^\dagger = \frac{\hat{x}_0 - i\hat{x}_1}{\sqrt{2\theta}} , \quad [a\,, a^\dagger] = 1\,, \tag{36}$$

and

$$|z\rangle = e^{\frac{1}{\sqrt{2\theta}}(za^\dagger - \bar{z}a)}|0\rangle\,, \tag{37}$$

where $a|z\rangle = \frac{z}{\sqrt{2\theta}}|z\rangle$ and $\langle z|z\rangle = 1$. We have

$$\langle z|\hat{x}_\mu|z\rangle = x_\mu\,, \tag{38}$$

where $z = x_0 + ix_1$.

We can now compute (35) using the resolution of identity

$$\mathbf{1} = \int \frac{d^2z'}{4\pi\theta}|z'\rangle\langle z'|\,. \tag{39}$$

In particular we find that

$$Tr\hat{\gamma} = \int \frac{d^2z'}{4\pi\theta}\langle z'|\hat{\gamma}|z'\rangle = \frac{T}{2\pi\theta}\,. \tag{40}$$

Appendix 1 contains the details.

Note that (40) diverges as $\theta \to 0$. That is because $\hat{\delta}(\hat{x}_1 - x_1)$ is not normalizable in the commutative limit just like a plane wave.

The object I of our interest is the intensity as measured by ω_γ:

$$I = \omega_{\hat{\gamma}}(|\hat{\psi}|^2)\,. \tag{41}$$

We find, using coherent states for example, that

$$I = \frac{Tr\hat{\gamma}|\hat{\psi}|^2}{Tr\hat{\gamma}} = \begin{cases} 2\left[1 + \left(1 - \frac{2\theta w}{T}\right)\cos 2w(x_1 - \theta w)\right] & \text{for} \quad 2\theta w < T\,, \\ 2 & \text{for} \quad 2\theta w \geq T \end{cases} \tag{42}$$

as Appendix 3 shows.

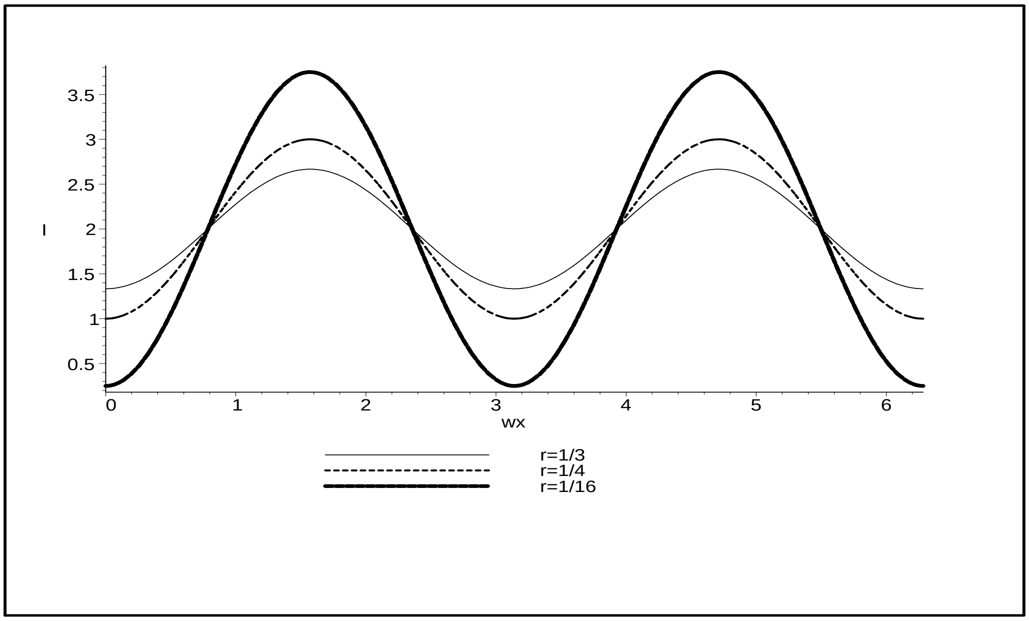

Figure 1: Variation of intensity I as a function of wx_1, for fixed $\frac{\theta w}{T} < \frac{1}{2}$. The plots are for $r = \frac{\theta w}{T} = \frac{1}{3}, \frac{1}{4}, \frac{1}{16}$ and $\theta w^2 = \frac{\pi}{2}$. Clearly, as the ratio $\frac{\theta w}{T}$ gets smaller, it converges to the commutative result.

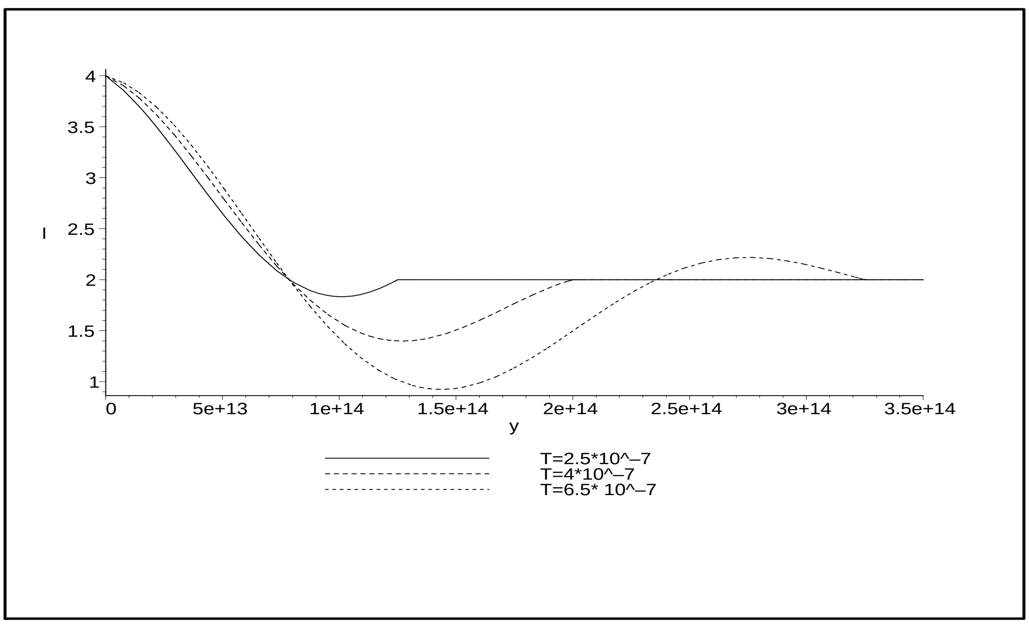

Figure 2: Variation of intensity I as a function of $y = \theta$. It is plotted at $x_1 = 0$, $\omega = 10^{-7}m^{-1}$ for the values of $T = 2.5 \times 10^7, 4 \times 10^7, 6.5 \times 10^7 m$. Note that for $\theta = 0$, $I = 4$ and for any given T, I takes the value 2 and becomes independent of θ at $\theta = \frac{T}{2} \times 10^7$ and thereon.

This result is remarkable. It asserts that *there is no interference at all if* $\frac{\theta w}{T} > \frac{1}{2}$*!* Thus the higher the frequency, the larger is the time of observation needed to perceive interference. There *is* interference for $\frac{\theta w}{T} < \frac{1}{2}$, but its pattern is shifted and extrema modified depending on frequency and time of observation. Figure 1 illustrates the phenomenon. We recover the usual pattern when $\frac{\theta w}{T} \to 0$, and in particular in the commutative limit. Variation of I w.r.t. $2\theta w$ is plotted in Figure 2.

b) $d = 3$:

We now consider the algebra $\mathcal{A}_\theta(\mathbb{R}^3)$. A good problem to study is Young's double slit experiment. For this, let us imagine a screen (a line) along the 2-direction. The two slits are a distance a apart and the line joining them is also parallel to 2-axis. This line is at a fixed distance from the screen.

As spatial coordinates commute, there is no intrinsic difficulty in realizing an arrangement like the above with arbitrary precision.

Let $\vec{n} = (1, 0)$. Then time-space non-commutativity is expressed as

$$[\hat{x}_0, \hat{x}_1] = i\theta\,, \quad [\hat{x}_0, \hat{x}_2] = 0\,. \tag{43}$$

The wave is considered to be associated with a massless scalar field. Then as in (23), a plane wave is

$$\hat{\psi}_1 = e^{i\mathbf{k}\cdot\hat{\mathbf{r}}} e^{-ik\hat{x}_0}\,, \quad \hat{\mathbf{r}} \equiv (\hat{x}_1\,, \hat{x}_2)\,, \quad \mathbf{k}\cdot\hat{\mathbf{r}} = k_i\hat{x}_i\,, \quad |\mathbf{k}| \equiv k = \sqrt{k_1^2 + k_2^2}\,. \tag{44}$$

We can also spatially translate this wave by $\vec{a}$ and evolve it for time τ by applying $e^{i\hat{\vec{P}}\cdot\vec{a}} e^{-i\hat{P}_0\tau}$, the result is still a plane wave:

$$e^{-i\hat{P}_0\tau} e^{i\hat{\vec{P}}\cdot\vec{a}} \big(e^{i\mathbf{k}\cdot\hat{\mathbf{r}}} e^{-ik\hat{x}_0}\big) = \big(e^{i\mathbf{k}\cdot\hat{\mathbf{r}}} e^{-ik\hat{x}_0}\big) e^{i\mathbf{k}\cdot\mathbf{a}} e^{-ik\tau}\,. \tag{45}$$

The last factor is the complex-valued phase of the wave. It lets us to unambiguously compare the phases of waves related by space-time translations. As the phase does not change if $\vec{a} = \frac{\vec{k}}{k}\tau$ for $k \neq 0$, the phase being zero for $k = 0$, we can say that the wave travels in the direction $\vec{k}$ as in the $\theta = 0$ limit.

We want to consider the interference of the waves from the two slits at a point P on the screen, assuming for simplicity that they have the same frequency w.

Let $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$ be the directions of propagation from the slits to P. Then the wave at P is

$$\begin{aligned} \hat{\psi} &= \hat{\psi}_1 + \hat{\psi}_2\,, \\ \hat{\psi}_1 = e^{i\mathbf{k}\cdot\hat{\mathbf{r}}} e^{-ik\hat{x}_0}\,, &\qquad \hat{\psi}_2 = e^{i\mathbf{k}'\cdot\hat{\mathbf{r}}'} e^{-ik\hat{x}_0}\,, \end{aligned} \tag{46}$$

where

$$\mathbf{k} = k\hat{\mathbf{k}}\,, \quad \mathbf{k}' = k'\hat{\mathbf{k}}'\,, \quad \hat{\mathbf{r}}' = \hat{\mathbf{r}} + \hat{\mathbf{a}} \tag{47}$$

Here $\mathbf{a} = a(0, 1)$ is the displacement of the primed slit relative to the other one.

Note that the waves do not acquire phases as they arrive at P from the slits a time τ later as the remark after (45) shows.

We generalize the density matrix $\hat{\gamma}$ according to

$$\widehat{\Gamma} = \hat{\psi}_T(\hat{x}_0 - x_0)\hat{\delta}(\hat{x}_1 - x_1)\hat{\hat{\delta}}(\hat{x}_2 - x_2)\hat{\psi}_T(\hat{x}_0 - x_0)\,, \tag{48}$$

where $\hat{\delta}(\hat{x}_1 - x_1)$ and $\hat{\psi}_T$ are given by (27) and (28) respectively, while $\hat{\hat{\delta}}(\hat{x}_2 - x_2)$ is a regularized version of the delta function centered at x_2(see below). Consequently, the state generalizing (25) is given by

$$\omega_{\widehat{\Gamma}}(\hat{\alpha}) = \frac{Tr\widehat{\Gamma}\hat{\alpha}}{Tr\widehat{\Gamma}}\,. \tag{49}$$

Then the intensity I due to $\hat{\psi}$ at P is given by $\omega_{\widehat{\Gamma}}(\hat{\psi}^\dagger\hat{\psi})$.

Note that the standard δ-function $\hat{\delta}(\hat{x}_2 - x_2)$ is not normalizable, having infinite trace. Hence, to regularize its contribution to the traces, we replace it, for example by $\hat{\hat{\delta}}(\hat{x}_2 - x_2) = |\alpha\rangle\langle\alpha|$ where

$$\langle x_2'|\alpha\rangle\langle\alpha|x_2'\rangle \tag{50}$$

is a Gaussian of width d centred at $x_2' = x_2$. For this purpose we introduce the momentum $\hat{P}_2$ conjugate to $\hat{x}_2$ with the eigenfunctions $e^{ip_2'(x_2'-x_2)}$ and eigenvalues $p_2' \in (-\infty\,,\infty)$. Consider

$$\begin{aligned}\langle x_2'|\alpha\rangle &= \frac{1}{\sqrt{2\pi}}\frac{\sqrt{d}}{\pi^{1/4}}\int dp_2' e^{ip_2'(x_2'-x_2)}e^{-\frac{1}{2}d^2p_2'^2}\\ &= \frac{1}{\pi^{1/4}\sqrt{d}}e^{-\frac{1}{2d^2}(x_2'-x_2)^2}\,.\end{aligned} \tag{51}$$

(51) will do the job: (50) is a Gaussian, which in the $d \to 0$ limit is a delta function centred at x_2, while $Tr|\alpha\rangle\langle\alpha| = 1$.

With $\hat{\hat{\delta}}(\hat{x}_2 - x_2) = |\alpha\rangle\langle\alpha|$, traces can easily be computed in the basis $|z'\rangle \otimes |x_2'\rangle$ ($|z'\rangle$ being the coherent states used in the previous section). We observe that $Tr\widehat{\Gamma} = \frac{T}{2\pi\theta}$, while for the intensity I we find

$$\begin{aligned}I = \omega_{\widehat{\Gamma}}(\hat{\psi}^\dagger\hat{\psi}) &= \lim_{d\to 0} 2\Big[1 + \Big(1 - \frac{\theta|k_1 - k_1'|}{T}\Big)e^{-\frac{1}{4}d^2(k_2-k_2')^2}\cos\Big((k_1 - k_1')(x_1 - k\theta)\\ &\qquad + (k_2 - k_2')x_2 - k_2'a\Big)\Big]\\ &= 2\Big[1 + \Big(1 - \frac{\theta|k_1 - k_1'|}{T}\Big)\cos\Big((k_1 - k_1')(x_1 - k\theta) + (k_2 - k_2')x_2 - k_2'a\Big)\Big]\\ &\qquad \text{for} \quad \theta|k_1 - k_1'| < T\,,\end{aligned}$$

$$I = 2 \quad \text{for} \quad \theta|k_1 - k_1'| \geq T\,. \tag{52}$$

Note that the final result is independent of d.

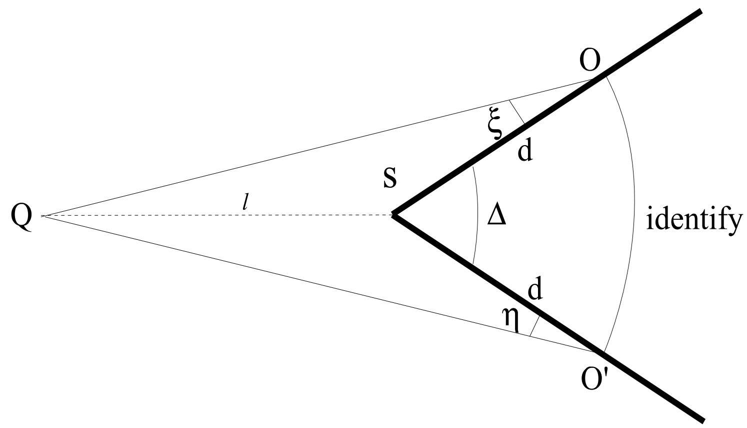

Figure 3: Double image of Q is observed by $O \equiv O'$.

This result is similar to the d =2 case. There is no interference at all for $\theta|k_1 - k_1'| \geq T$ and the interference pattern is distorted for $\theta|k_1 - k_1'| < T$ in a similar manner encountered in (42).

When $\theta|k_1 - k_1'| < T$, the dependence of I on $(k_1 - k_1')x_1$ can be plotted for fixed values of $\frac{\theta|k_1 - k_1'|}{T}$. With suitable choice of the phases in (52), the plots will look similar to those in Figure 1.

c) d =4 : Interference Phenomena from Cosmic Strings

As an example we here study interference of waves from a distant source caused by a cosmic string. For simplicity we consider a straight cosmic string. The metric around it is flat with deficit angle $8\pi G\mu$, where μ is the mass per unit length of the string and G is the gravitational constant.

Figure 3 shows a spatial slice where Q is the source and S is the string. We assume that S is normal to the spatial slice as it is sufficient for our purposes. The string causes deficit angles and requires the straight lines SO and SO' to be identified. This identification causes a double image of Q. $\Delta = \widehat{OSO'}$ is the deficit angle $8\pi G\mu$. Q, S, O and O' are on a plane $\mathcal{P}$. Calling l the distance from the string to the source and d the distance from the observer to the string, the angular separation of the images is [13]

$$\Xi = \xi + \eta = \frac{l}{l+d}\Delta\,, \tag{53}$$

(where ξ and η are explained by Figure 3.)

We note that all the spatial coordinates commute with each other. Hence for the set-up above there is no intrinsic difficulty in defining the spatial directions and their orthogonality. They are exactly the same as in the commutative case.

It should be clear that the problem is effectively of two spatial dimensions. On the spatial slice any fixed vector from the string can be taken as the direction noncommuting with the time coordiante.

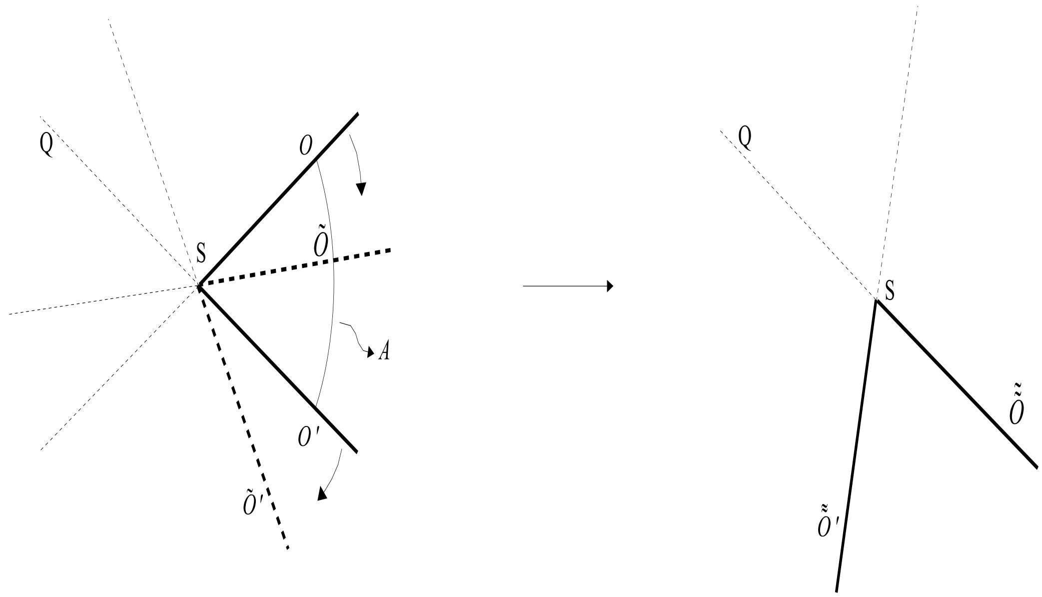

Figure 4: The observer ε at $O \equiv O'$ can barely observe a double image of the source Q. Keeping her distance from the string fixed, she can shift her position along the arc A and can observe a double image of Q from $\tilde{O} \equiv \tilde{O}'$. At $\tilde{\tilde{O}} \equiv \tilde{\tilde{O}}'$ she again can only barely observe a double image of Q.

Consider now the identified observers at O and O'. Call her ε. It is not hard to see from Figure 4 that keeping her distance d from the string S fixed, she can observe double images due to Q on all points on an arc A of length $d\Delta$. (See Figure 4).

Suppose now that she can observe the intensity of waves of definite frequency w coming from Q. Let us denote the wave vectors of the two plane waves $\hat{\psi}_1$ and $\hat{\psi}_2$ emerging from Q that reach the observer ε at $O \equiv O'$ by $\mathbf{k}$ and $\mathbf{k}'$, respectively. At $O \equiv O'$ the angle between $\mathbf{k}$ and $\mathbf{k}'$ is equal to Ξ. The intensity observed by ε at $O \equiv O'$ is then given by (52), where a in that formula (the separation between the slits) is now zero as there is only a single source Q. The observer ε can shift the observed intensity by shifting her position on the arc A. Thus we can think of A as a screen, where the interference pattern is recorded.

After some trigonometry we find that (see Appendix 4)

$$|k_1 - k_1'| = \left|(k_2 + k_2') \tan \frac{\Xi}{2}\right|. \tag{54}$$

From this result it is possible to estimate an upper bound on $\frac{\theta}{T}$ in order that the two waves interfere. Recall that for this we must have $\frac{\theta|k_1 - k_1'|}{T} < 1$. Substituting for $|k_1 - k_1'|$ from (54), we get $\frac{\theta|k_2+k_2'|}{T} < \left|\cot \frac{\Xi}{2}\right|$. It is known that double images of quasars are usually separated by a few arc seconds [14]. Taking for example an angular separation of 5 arc seconds puts the bound $\frac{\theta|k_2+k_2'|}{T} < 8.25 \times 10^4$. We note that $|k_2 + k_2'| = \alpha k$ where $0 \leq \alpha < 2$. Therefore, $\frac{\theta}{T} < \frac{8.25\times 10^4}{\alpha k}$ and for a given k the lowest upper bound will be approached as

$\alpha \to 2$. Finally, suppose that a wavelength at the red end of the visible spectrum is observed, say at $\lambda = \frac{2\pi}{k} = 700nm$. Then we find $\frac{\theta}{T} < 4.6 \times 10^{-3}m$. Only under this condition is the interference observable for light with wavelength $\lambda = 700nm$.

3 Conclusions

In this work we have studied the general theory of waves in Groenewold-Moyal spacetimes where time and space coordinates do not commute. We have given the rules for the measurement of their intensity and applied them to interference and diffraction phenomena in spacetimes of dimensions $d \leq 4$. The latter produced novel physical results. Namely, we found out that for observation times T which are so brief that $T \leq 2\theta w$, no interference can be observed. For larger times, the interference pattern is deformed and depends on $\frac{\theta w}{T}$. It approaches the commutative pattern only when $\frac{\theta w}{T} \to 0$. These results are given concretely by the equations (42) and (52) for $d = 2$ and $d = 3$, respectively. Finally, we have used these results to discuss the interference of stellar light due to cosmic strings, where with the help of the present stellar data we have estimated that for a given k, we must have $\frac{\theta}{T} \lesssim \frac{4\times 10^4}{k}$ to observe interference.

Acknowledgments

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Appendix 1

In this Appendix we calculate $Tr\hat{\gamma}$ where $\hat{\gamma}$ is given by (26). In the coherent state basis $|z'\rangle$ we have

$$\begin{aligned} Tr\hat{\gamma} &= \int \frac{d^2z'}{4\pi\theta} \langle z'|\hat{\gamma}|z'\rangle \\ &= \int \frac{d^2z'}{4\pi\theta} \langle z'|\hat{\psi}_T(\hat{x}_0 - x_0)\hat{\delta}(\hat{x}_1 - x_1)\hat{\psi}_T(\hat{x}_0 - x_0)|z'\rangle. \end{aligned} \tag{A.1}$$

Using (27) and (28) in (A.1), we get

$$
\begin{aligned}
Tr\hat{\gamma} &= \int d\lambda_1 d\lambda_2 \; Tr \; \hat{\delta}(\hat{x}_0 - x_0 - \lambda_1) \; \hat{\delta}(\hat{x}_1 - x_1)\hat{\delta}(\hat{x}_0 - x_0 - \lambda_2) \\
&= \frac{1}{(2\pi)^3} \int d\lambda_1 d\lambda_2 dk_1 dk_2 dk_3 \; Tr \; e^{ik_1(\hat{x}_0 - x_0 - \lambda_1)} e^{ik_2(\hat{x}_1 - x_1)} e^{ik_3(\hat{x}_0 - x_0 - \lambda_2)} \qquad \text{(A.2)} \\
&= \frac{1}{(2\pi)^3} \int d\lambda_1 d\lambda_2 dk_1 dk_2 dk_3 e^{i[\theta k_2 k_3 - k_1(x_0+\lambda_1) - k_2 x_1 - k_3(x_0+\lambda_2)]} Tr e^{i(k_1+k_3)\hat{x}_0} e^{ik_2\hat{x}_1}
\end{aligned}
$$

Using (36) and in the coherent state basis we have

$$
\begin{aligned}
Tr e^{i(k_1+k_3)\hat{x}_0} e^{ik_2\hat{x}_1} &= \langle z' | e^{i(k_1+k_3)\hat{x}_0} e^{ik_2\hat{x}_1} | z' \rangle \\
&= \int \frac{dx'_0 dx'_1}{2\pi\theta} e^{[\frac{\theta}{4}\{(k_1+k_3)^2 + k_2^2\} - i\frac{\theta}{2}(k_1+k_3)k_2]} e^{i(k_1+k_3)x'_0} e^{ik_2 x'_1} \\
&= \frac{2\pi}{\theta} e^{[\frac{\theta}{4}\{(k_1+k_3)^2 + k_2^2\} - i\frac{\theta}{2}(k_1+k_3)k_2]} \delta(k_1+k_3) \; \delta(k_2) \\
&= \frac{2\pi}{\theta} \delta(k_1+k_3) \; \delta(k_2), \qquad \text{(A.3)}
\end{aligned}
$$

where we have used $z' = x'_0 + ix'_1$ and $d^2z' = 2dx'_0 dx'_1$. Substituting (A.3)in (A.2) and remembering that $-\frac{T}{2} \leq \lambda_i \leq \frac{T}{2}, \quad i = 1, 2$, we get

$$
\begin{aligned}
Tr\hat{\gamma} &= \frac{1}{4\pi^2\theta} \int d\lambda_1 d\lambda_2 dk_3 e^{ik_3(\lambda_1+\lambda_2)} \\
&= \frac{1}{2\pi\theta} \int d\lambda_1 d\lambda_2 \delta(\lambda_1+\lambda_2) \\
&= \frac{T}{2\pi\theta} \qquad \text{(A.4)}
\end{aligned}
$$

Appendix 2

In this Appendix we give the derivation of (35). Using (26), (27), (28) and (33) we get

$$
\begin{aligned}
Tr\hat{\gamma}\hat{\alpha} &= Tr\hat{\psi}_T(\hat{x}_0 - x_0)\hat{\delta}(\hat{x}_1 - x_1)\hat{\psi}_T(\hat{x}_0 - x_0)\hat{\delta}(\hat{x}_1 - y_1) \\
&= \frac{1}{(2\pi)^4} \int d\mu \; Tr e^{ik_1(\hat{x}_0 - x_0 - \lambda_1)} e^{ik_2(\hat{x}_1 - x_1)} e^{ik_1(\hat{x}_0 - x_0 - \lambda_2)} e^{ik_2(\hat{x}_1 - y_1)} \qquad \text{(A.5)} \\
&= \frac{1}{(2\pi)^4} \int d\mu \; e^{i[\theta k_2 k_3 - k_1(x_0+\lambda_1) - k_2 x_1 - k_3(x_0+\lambda_2) - k_4 y]} \; Tr e^{i(k_1+k_3)\hat{x}_0} e^{i(k_2+k_4)\hat{x}_1},
\end{aligned}
$$

where $d\mu = d\lambda_1 d\lambda_2 dk_1 dk_2 dk_3 dk_4$ and $-\frac{T}{2} \leq \lambda_i \leq \frac{T}{2}, \quad i = 1, 2$. Using (36) and in the coherent state basis we get

$$
\begin{aligned}
Tr e^{i(k_1+k_3)\hat{x}_0} e^{i(k_2+k_4)\hat{x}_1} &= \frac{2\pi}{\theta} e^{\frac{\theta}{4}[(k_1+k_3)^2 + (k_2+k_4)^2]} \delta(k_1+k_3)\delta(k_2+k_4) \\
&= \frac{2\pi}{\theta} \delta(k_1+k_3)\delta(k_2+k_4) \qquad \text{(A.6)}
\end{aligned}
$$

where we have used a technique similar to that was used to derive (A.3). Using (A.6) in (A.5) we get

$$\begin{aligned} Tr\hat{\gamma}\hat{\alpha} &= \frac{1}{(2\pi\theta)^2}\int d\lambda_1 d\lambda_2\ e^{i\lambda_1\frac{(y_1-x_1)}{\theta}}e^{i\lambda_2\frac{(y_1-x_1)}{\theta}} \\ &= \frac{1}{\pi^2(x_1-y_1)^2}\ \sin^2\left[\frac{T(x_1-y_1)}{2\theta}\right] \end{aligned} \tag{A.7}$$

Using (61) and (58) in (25) we get

$$\omega_\gamma(\hat{\alpha}) = \frac{2}{\pi}\frac{\theta}{T}\frac{1}{(x_1-y_1)^2}\sin^2\left[\frac{T(x_1-y_1)}{2\theta}\right]. \tag{A.8}$$

Appendix 3

In this Appendix we provide the derivation of the intensity formula (42). Using (23), we get

$$\begin{aligned} Tr\hat{\gamma}|\hat{\psi}|^2 &= 2Tr\hat{\gamma} + e^{-2ik^2\theta}Tr\hat{\gamma}e^{2ik\hat{x}_1} + e^{2ik^2\theta}Tr\hat{\gamma}e^{-2ik\hat{x}_1} \\ &= 2\left[Tr\hat{\gamma} + \mathrm{Re}(e^{-2ik^2\theta}Tr\hat{\gamma}e^{2ik\hat{x}_1})\right]. \end{aligned} \tag{A.9}$$

Substituting for $\hat{\gamma}$ from (26), we can write

$$Tr\hat{\gamma}e^{2ik\hat{x}_1} = \frac{1}{(2\pi)^3}\int d\mu e^{-i[k_2k_3\theta+k_1(x_0+\lambda_1)+k_2x_1+k_3(x_0+\lambda_2)]}\ Tre^{i(k_1+k_3)\hat{x}_0}e^{i(k_2+2k)\hat{x}_1}, \tag{A.10}$$

where $d\mu = d\lambda_1 d\lambda_2 dk_1 dk_2 dk_3$. Using (36) and in the coherent state basis we get

$$\begin{aligned} Tre^{i(k_1+k_3)\hat{x}_0}e^{i(k_2+2k)\hat{x}_1} &= \frac{2\pi}{\theta}e^{\frac{\theta}{4}[(k_1+k_3)^2-(k_2+2k)^2]}\delta(k_1+k_3)\delta(k_2+2k) \\ &= \frac{2\pi}{\theta}\delta(k_1+k_3)\delta(k_2+2k). \end{aligned} \tag{A.11}$$

Hence

$$\begin{aligned} e^{-2ik^2\theta}Tr\hat{\gamma}e^{2ik\hat{x}_1} &= \frac{e^{2i(kx_1-k^2\theta)}}{(2\pi)\theta}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\lambda_2\int_{-\frac{T}{2}}^{\frac{T}{2}} d\lambda_1\ \delta(\lambda_1-\lambda_2-2\theta k) \\ &= \frac{e^{2i(kx_1-k^2\theta)}}{(2\pi)\theta}\int_{-\frac{T}{2}}^{\frac{T}{2}-2k\theta} d\lambda_2. \end{aligned} \tag{A.12}$$

Remembering that $\theta k > 0$, we see that the integrand in (A.12) vanishes if $2\theta k > T$. If $2\theta k < T$ instead, we have

$$\begin{aligned} e^{-2ik^2\theta}Tr\hat{\gamma}e^{2ik\hat{x}_1} &= \frac{e^{2i(kx_1-k^2\theta)}}{(2\pi)\theta}\int_{-\frac{T}{2}}^{\frac{T}{2}-2k\theta} d\lambda_2 \\ &= \frac{e^{2i(kx_1-k^2\theta)}}{\pi\theta}\left(\frac{T}{2}-\theta k\right). \end{aligned} \tag{A.13}$$

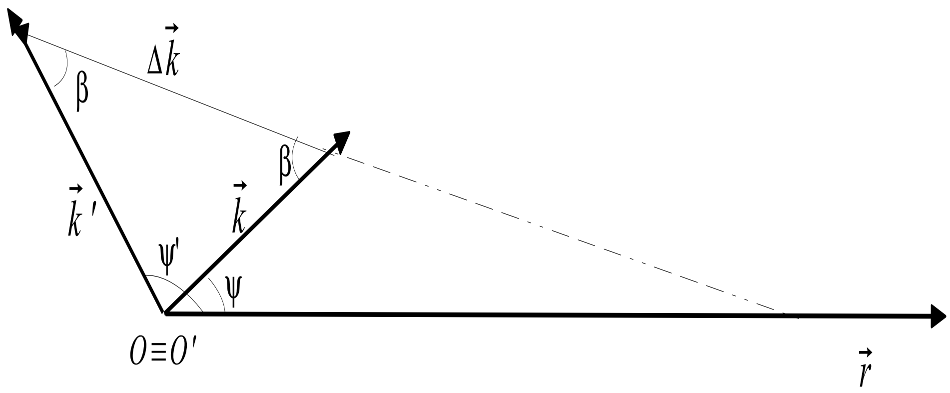

Figure A. 1: The diagram provides the definition of angles $\psi\,,\psi'$ and β.

From (A.9), (A.12), (A.13) and (A.4) we finally get

$$I=\frac{Tr\hat{\gamma}|\hat{\psi}|^2}{Tr\hat{\gamma}}=\begin{cases} 2\left[1+\left(1-\frac{2\theta w}{T}\right)\cos 2w(x_1-\theta w)\right] & \text{for} \quad 2\theta w<T\,, \\ 2 & \text{for} \quad 2\theta w\geq T \end{cases}\,, \tag{A.14}$$

where we have used $w=k$.

Appendix 4

In this appendix we give a derivation of the result (54). From the figure above we see the following relations between the angles

$$\psi'-\psi=\Xi\,, \tag{A.15}$$

$$2\beta+\Xi=180 \quad\Rightarrow\quad \beta=90-\frac{\Xi}{2}\,, \tag{A.16}$$

$$2\beta+\psi'-\psi=180 \quad\Rightarrow\quad \psi'+\beta=180-\beta+\psi\,. \tag{A.17}$$

Noting that $\cos\psi=\frac{\mathbf{k}\cdot\mathbf{r}}{kr}\,,\sin\psi=\frac{\hat{n}.(\mathbf{k}\times\mathbf{r})}{kr}$ ($\hat{n}$ being the unit normal in the direction of $\mathbf{k}\times\mathbf{r}$) and $|k|=|k'|=k$, we explicitly have

$$\cos\psi=\frac{1}{k}\left(k_1\frac{x_1}{r}+k_2\frac{x_2}{r}\right) \quad \sin\psi=\frac{1}{k}\left(k_1\frac{x_2}{r}-k_2\frac{x_1}{r}\right)\,, \tag{A.18}$$

and similarly

$$\cos\psi'=\frac{1}{k}\left(k'_1\frac{x_1}{r}+k'_2\frac{x_2}{r}\right) \quad \sin\psi'=\frac{1}{k}\left(k'_1\frac{x_2}{r}-k'_2\frac{x_1}{r}\right)\,. \tag{A.19}$$

Using the above relations we find that

$$\begin{aligned}\cos(\beta+\psi') &= \frac{1}{k}\left[\left(k_1'\sin\frac{\Xi}{2}+k_2'\cos\frac{\Xi}{2}\right)\frac{x_1}{r}+\left(k_2'\sin\frac{\Xi}{2}-k_1'\cos\frac{\Xi}{2}\right)\frac{x_2}{r}\right]\,,\\ \cos(180-\beta+\psi) &= \frac{1}{k}\left[\left(-k_1\sin\frac{\Xi}{2}+k_2\cos\frac{\Xi}{2}\right)\frac{x_1}{r}+\left(-k_2\sin\frac{\Xi}{2}-k_1\cos\frac{\Xi}{2}\right)\frac{x_2}{r}\right]\,.\end{aligned} \tag{A.20}$$

Due to (A.17) the difference of the the cosines above vanish. For this to happen, we observe that the coefficients of $\frac{x_1}{r}$ and $\frac{x_2}{r}$ must separately vanish. Thus the coefficients of $\frac{x_2}{r}$ must satisfy

$$(k_2+k_2')\sin\frac{\Xi}{2} = (k_1'-k_1)\cos\frac{\Xi}{2}\,, \tag{A.21}$$

and hence (54).

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[14] See for example, CASTLES Survey Data, at the web site: http://cfa-www.harvard.edu/castles/.